The Paradox of Voting and Candidate Competition: A General Equilibrium Analysis

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Conventional analysis of the decision of expected utility maximizing agents to vote has concluded that it is irrational to vote unless voters have a distorted view of their individual impact or place a direct value on the act of voting.[1] On the other hand, mathematical analyses of the electoral process (see, for example, Davis, Hinich, and Ordeshook, 1970) have usually assumed that all voters vote.[2] Each theory is incorrect, in the sense that in actual elections turnout is neither zero nor 100 percent.

In this paper we will argue that previous analyses of expected utility maximizing voters stopped too soon, because of the partial equilibrium approach, and that if each voter considers the simultaneous reactions of all voters in a "rational" manner, then, depending on the location of the candidates' platforms, turnout will usually be positive but less than 100 percent. In particular, we will derive a (probabilistic) vote supply function, given a distribution of voters and the choice of platforms of candidates, which has the property that—even with costs of voting (unless the candidates have identical platforms)—the expected turnout is positive. The model and these results are presented in sections IA and IB.

Since Ferejohn and Fiorina (1974 and 1975) have presented persuasive theoretical and empirical arguments that another form of rational behavior, "minimax regret," is realistic and has the property that turnout is positive (unless platforms are identical), we will spend some time comparing the implications of their model and ours. Essentially, we claim that our model predicts larger turnout than their model when preferences (ideal points) are symmetrically distributed, candidates' platforms are close, and the variance of tastes is small (with concave utility functions) or large (with a type of convex utility function). Predicted turnout is larger in their model for opposite values. Thus, neither dominates the other with respect to predicted turnout. This is discussed in detail in section IC.

The results concerning turnout and voter behavior in both models depend on the candidates' choices of their platforms. Thus a natural question is, What will candidates do, given our model of voters? This furthers the move to a general equilibrium approach since candidates' and voters' be-

havior now are simultaneously determined. Using expected plurality maximizing behavior for two-candidate elections, we obtained mixed results in our investigation. For concave utility functions and symmetrically distributed ideal points, equilibrium occurs with both candidates' choosing the median ideal point and no voter voting (since platforms are identical). Further, if ideal points are asymmetrically distributed in the tails of the distributions (that is, at extreme distances from the median), usually no general equilibrium will exist. These results appear in sections IIA, IIB(i), and IIB(ii).

On the other hand, if utility functions are convex on each side of the ideal point and if tastes are unimodal and not too asymmetric, a general equilibrium exists with candidates choosing identical platforms at the *modal* ideal point and no voter voting. These results are included in section IIB(iii).

A summary of results is provided in section IIC, with some additional remarks. One deserves emphasis. In general equilibrium, with expected plurality maximizing candidates, the outcome is identical for two models of voter behavior: ours and minimax regret. That is, both models predict, in equilibrium, identical candidate platforms and no voter turnout if costs of voting are positive.

In section III we consider three additional problems: (a) the implications of vote maximizing candidates—turnout is positive in equilibrium if it exists; (b) M candidate elections for $M \geq 2$, although no general equilibrium results are presented where candidate behavior is included; and (c) some remarks on testing models of simultaneous voter and candidate behavior—particularly our model. This section concludes the paper.

I. VOTER (PARTIAL) EQUILIBRIUM: TWO CANDIDATES

We begin with the conventional analysis of the voting decision of a single voter in the spirit of Downs, Tullock, Riker-Ordeshook, and Ferejohn and Fiorina. In this model, candidates A and B select platforms $\Theta_A, \Theta_B \in H$, an "issue space." A voter than votes for (say) A, if and only if the expected utility outweighs the expected utility from voting for B or from abstaining.

A. The Conventional Analysis

We consider a model with $n + 1$ voters, indexed by $i = 1, 2, \ldots, n + 1$. Each voter has preferences over a set of possible issues, H. Two candidates are indexed $j = A, B$, where $\Theta_j \in H$ is j's platform. When a voter votes for a particular candidate (or abstains), he is implicitly selecting a gamble, since at the time of the decision he does not know how other voters will vote. We make the standard assumption that the voter who makes the decision under uncertainty acts as if he maximizes expected utility.

ASSUMPTION 1. With each voter i is associated a utility function U^i on H, such that

(a) (candidate irrelevance) i prefers candidate A to B if and only if $U^i(\Theta_A) > U^i(\Theta_B)$, and

(b) (expected utility hypothesis) letting $(\Pi_A, \Pi_B, \Theta_A, \Theta_B)$ represent the gamble that j is elected on platform Θ_j with probability Π_j (for $j = A, B$), voter i prefers $(\Pi_A, \Pi_B, \Theta_A, \Theta_B)$ to $(\Pi'_A, \Pi'_B, \Theta'_A, \Theta'_B)$ if and only if $\Pi_A \cdot U^i(\Theta_A) + \Pi_B \cdot U^i(\Theta_B) > \Pi'_A \cdot U^i(\Theta'_A) + \Pi'_B \cdot U^i(\Theta'_B)$.

To ease the exposition and pave the way for later analysis, we make the additional assumption that each voter's (expected) utility function can be parameterized. That is, we let D be a space of voters' characteristics and let $U^i(\Theta) = U(\Theta, d^i)$ be the utility of i for Θ if his characteristic is $d^i \in D$. Three simple examples may help the reader understand the notation.

Example 1 (Type I preferences).[3] Let $H \equiv R^L$, the L dimensional Euclidean space. Let $D \equiv R^L$ and let $U(\Theta, d^i) = -(\Theta - d^i)'(\Theta - d^i) = -\Sigma_{\ell=1}^{L}(\Theta_\ell - d^i_\ell)^2$. Thus, every voter has a quadratic utility (loss) function, whose ideal point is d^i, over L issues measured as real numbers.

Example 2 (Simple social choice). Let $H = \{x_1, x_2, x_3\}$. That is, there are only three alternatives. Let $D \equiv R^3$ and let $U(x_k, d^i) = d^i_k$ for $k = 1,2,3$. Thus $d^i = (d^i_1, d^i_2, d^i_3)$, where d^i_k is i's utility for alternative k.

Example 3 (Type II preferences). Let $H \equiv R^L$, $D = R^L$ and $U(\Theta, d^i) = -[(\Theta - d^i)'(\Theta - d^i)]^{\frac{1}{m}}$, $m > 1$. As for type I preferences, each voter has an ideal point d^i. However, while type I preferences are concave utility functions, type II preferences are convex, on each side of d^i. As we will see, the behavior implied by type II preferences is significantly different from behavior implied by type I preferences.

A digression. Assumption 1(a) can be weakened, in what follows, to allow voter identification of candidates to be important. For example, suppose i believes *ex ante* that if j adopts the platform Θ_j, then j will implement the platform γ, if elected, with probability $\ell^j(\gamma, \Theta_j, d^i)$. Then i's *ex ante* utility for j, given the platform Θ_j, is $W^j(\Theta_j, d^i) = \int U(\gamma, d^i)\ell^j(\gamma, \Theta_j, d^i)d\gamma$. Thus, i prefers A to B if and only if $W^A(\Theta_A, d^i) > W^B(\Theta_B, d^i)$, and even if $\Theta_A = \Theta_B$, i may prefer A to B. Since this does not explain where the $\ell^i(\cdot)$ likelihood functions come from, and since this generality tends to obscure the main issues, we will reconsider it only if it has some significant bearing on the results to be derived.

One characteristic (in addition to d^i) which we also need to consider is the cost of voting, c^i. We will assume that $0 < \underline{c} \le c^i < \infty$ for all i; that is, all voters must incur a cost if they vote, and these costs are bounded away from zero by $\underline{c}$. For shorthand purposes only, we will let $e^i = (d^i, c^i)$ and $E^i \equiv D \times (\underline{c}, \infty)$.

ASSUMPTION 2 (No income effects). If candidate j wins, then voter i, with characteristic e^i, receives utility $U(\Theta_j, d^i) - c^i$ if he votes and $U(\Theta_j, d^i)$ if he abstains.

Another digression. Assumption 2 could be weakened to i receives $U(\Theta_j, c^i, d^i)$ if i votes and j wins, while i receives $U(\Theta_j, 0, d^i)$ if j wins and i abstains. Unfortunately, this complicates the analysis somewhat. Further, as far as I can tell, this weakening does not seem to alter the equilibrium results below. I will therefore stay with assumption 2 to ease exposition and to remain as close as possible to the standard framework.

We are now ready to analyze the voters' decision. A voter has three possible acts: vote for A, vote for B, or abstain. There are, essentially, five states of the world which must be considered. Let n_j be the number of votes cast by the other n voters for $j = A, B$. (Since we allow abstentions, $n_A + n_B < n$ is possible.) The five states are S_1, where $n_A > n_B + 1$; S_2, where $n_A = n_B + 1$; S_3, where $n_A = n_B$; S_4, where $n_B = n_A + 1$; and S_5, where $n_B > n_A + 1$. Let p^i be the probability of state k from i's point of view.

Lemma 1 (Ferejohn and Fiorina, 1974). If tied elections are decided by a fair coin toss, then, given Θ_A, Θ_B, e^i, and $p^i = (p^i_1, \ldots, p^i_5)$, voter i maximizes expected utility by (deleting the i on p^i_k):

$$\text{voting for } A \text{ if } W(\Theta, e^i) > \frac{1}{p_3 + p_4} \tag{1a}$$

$$\text{voting for } B \text{ if } -\frac{1}{p_2 + p_3} > W(\Theta, e^i) \tag{1b}$$

$$\text{abstaining if } -\frac{1}{p_2 + p_3} < W(\Theta, e^i) < \frac{1}{p_3 + p_4} \tag{1c}$$

$$\text{where } W(\Theta, e^i) \equiv \frac{U(\Theta_A, d^i) - U(\Theta_B, d^i)}{2c^i}.$$

For precision, the boundary cases in Lemma 1, when $W(\Theta, e^i) = \frac{1}{p_3 + p_4}$ or $W(\Theta, e^i) = -\frac{1}{p_2 + p_3}$, should be dealt with. At these values, i is indifferent between voting and abstaining. Thus one should make some assumption about the actual act chosen. Fortunately, this boundary situation will usually occur (below) with probability zero and may safely be ignored. If not, we will point out the implications at the appropriate time.

At this point, the conventional analysis notes that both $p_3 + p_4$ and $p_2 + p_3$ are objectively very small and, unless voters inflate their estimates or receive a direct utility gain from voting, a rational expected utility maximizing citizen will decide to abstain. This contradicts empirical evidence, since people *do* vote. As a solution to this apparent dilemma, Ferejohn and Fiorina suggest that instead of maximizing expected utility, voters act according to Savage's minimax regret criterion. It is useful for later analysis to summarize their results in our notation:

Lemma 2 (Ferejohn and Fiorina, 1974). If tied elections are decided by a fair coin toss, then, given Θ_A, Θ_B and e^i, voter i minimizes his maximum regret by:

$$\text{voting for } A \text{ if } W(\Theta, e^i) > 2 \tag{2a}$$

$$\text{voting for } B \text{ if } W(\Theta, e^i) < -2 \tag{2b}$$
$$\text{abstaining if } -2 < W(\Theta, e^i) < 2. \tag{2c}$$

We note that this is equivalent to expected utility maximizing if and only if

$$p_2 + p_3 = \frac{1}{2} = p_3 + p_4.$$

B. Full Rationality

In this section we consider an alternative to both models discussed in IA. In particular, we propose and analyze a solution to the paradox of voting suggested by Ferejohn and Fiorina but never followed up.[4] The solution is brought about by assuming that each voter is rational and that each assumes the others also are rational. This will allow us to calculate precisely what $p_2^i + p_3^i$ and $p_3^i + p_4^i$ are in the mind of each voter i. Further, we will be able to make some statements about expected turnout (which will, in general, be non-zero).

ASSUMPTION 3. (a: Each voter assumes all voters are rational). Each voter i believes all other voters follow the (expected utility maximizing) decision rules in lemma 1. (b: Independent-identical beliefs). Each voter i believes other voters' characteristics are independently and identically distributed on E, according to the probability measure μ.

Thus, although i doesn't know h's characteristic and, therefore, doesn't know how h will vote, he does know how h will vote (or abstain) if h has the characteristic e^h. He also believes that e^h is a random variable drawn from μ. For full rationality (as in a rational expectations equilibrium), one might want to assume that μ was the empirical distribution. Below, it will be helpful to have μ "continuous," and, thus, we usually assume that the true distribution of characteristics is approximated by a continuous density function. For large electorates this is not a severe limitation.

Another digression. The assumption of independent and identical beliefs is not crucial for much of what follows but *does* allow for considerable simplification of the analysis. We could replace A3 with the following weaker expectations hypothesis. Let $z^i = [d^1, \ldots, d^{i-1}, d^{i+1}, \ldots, d^n]$ and assume each i believes the others' characteristics, z^i, are distributed according to the measure $\Psi^i(e^i)(z^i)$ if i's characteristic is e^i. In this case, i's expectations can depend on e^i, whereas in assumption 3 they are independent of e^i. This is followed up in section IIIB.

Lemma 3. Under assumptions 1–3, given Θ_A, Θ_B, μ, and p^i, voter i believes the probability q_j, that an arbitrary voter h ($h \neq i$) votes for $j = 0, A, B$ ($j = 0$ means abstention), is (assuming $p^i = p^h$):

$$q_A = 1 - G(\frac{1}{\alpha}, \Theta_A, \Theta_B) \tag{3a}$$

$$q_B = G(-\frac{1}{\beta}, \Theta_A, \Theta_B) \tag{3b}$$

$$q_o = G(\frac{1}{\alpha}, \Theta_A, \Theta_B) - G(-\frac{1}{\beta}, \Theta_A, \Theta_B) \tag{3c}$$

where $\alpha = p_3^i + p_4^i$, $\beta = p_2^i + p_3^i$, and $G(r, \Theta_A, \Theta_B) = \mu(\{e^i \varepsilon E \mid W(\Theta_A, \Theta_B, e^i) \leq r\})$ [assuming that $\mu(\{e^i \varepsilon E \mid W(\Theta, e^i) = \frac{1}{\alpha}, W(\Theta, e^i) = -\frac{1}{\beta}\}) = 0$].

Proof. Straightforward application of assumption 3 and lemma 1.

Notice that if μ is concentrated on a finite number of points (say, $n + 1$) then the last qualifying phrase, needed for the case of indifference between voting and abstaining, may be false. We will assume shortly that μ, Θ_A and Θ_B are such that G is continuous in r. This will rule out μ concentrated on a finite number of characteristics and make the qualifying clause unnecessary.

Now if voter i knows q_A, q_B, and q_o, he is in a position to calculate $\alpha = p_3 + p_4$ and $\beta = p_2 + p_3$.

Lemma 4. Given q_A, q_B, and q_o,

$$\alpha = f(q_A, q_B) \tag{4a}$$
$$\beta = f(q_B, q_A) \tag{4b}$$

where $f(x,y) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{k}\binom{n-k}{k} x^k y^k (1-x-y)^{n-2k} + \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{k}\binom{n-k}{k+1} x^k y^{k+1} (1-x-y)^{n-2k-1}$, and $[v]$ is the largest integer that is no greater than v.

Proof. For n voters, let $(n_A, n_B, n-n_A-n_B)$ be the event where n_A vote for A, n_B vote for B, and $n-n_A-n_B$ abstain. Given q_A, q_B, and q_o, plus the independence assumption, the probability of $(n_A, n_B, n-n_A-n_B)$ is calculated to be $\binom{n}{n_A}\binom{n-n_A}{n_B}(q_A)^{n_A}(q_B)^{n_B}(1-q_A-q_B)^{n-n_A-n_B}$ from the trinomial distribution. The rest follows easily.

At this point it can be seen that q_A, q_B, α, and β are simultaneously determined and that all voters' decisions (as described in [1]) and expectations will be consistent and in equilibrium if, and only if, (3) and (4) are jointly satisfied.

Definition. A (symmetric) *voters' equilibrium* for $(\Theta_A, \Theta_B, \mu)$ is a 4-tuple $(q_A^*, q_B^*, \alpha^*, \beta^*)$, such that (3a), (3b), (4a), and (4b) are simultaneously satisfied.

Remarks. (1) The qualifier "symmetric" refers to the fact that all voters are assumed to have identical decision rules, (1), and identical expectations.

(2) This concept of equilibrium is a special case of a Bayes equilibrium in strategies, S^i; $E \rightarrow \{0, A, B\}$, where, for each e^i, $S^i(e^i)$ maximizes i's conditional expected utility, given the strategies $(S^1, \ldots, S^{i-1}, S^{i+1}, \ldots, S^n)$ of the other voters. This is explored more fully in section 3b.

Of interest, of course, is whether a voters' equilibrium exists and what its properties are. The second question is more difficult, primarily because of the cumbersome form of $f(x,y)$ in lemma 4, but as will be seen, we can say some things about it. The first is easy and so we turn to it now. We state a simple and somewhat uninteresting result.

Proposition 1. Given $(\Theta_A, \Theta_B, \mu)$, such that $G(1, \Theta_A, \Theta_B) - G(-1, \Theta_A, \Theta_B) = 1$, $(q^*_A, q^*_B, \alpha^*, \beta^*) = (0, 0, 1, 1)$ is the unique symmetric voters equilibrium.

Proof (Existence). One can easily show by substitution into (3) and (4) that $(0, 0, 1, 1)$ is an equilibrium under the assumption on G.

(Uniqueness). Under the assumption on G, it follows from (3) that $q_A = q_B = 0$ for any values of $\alpha, \beta \in [0,1]$. Thus by (4), $\alpha = \beta = 1$ must hold.

Remarks. Several remarks are in order. First, if $\Theta_A = \Theta_B$, then $G(1, \Theta) - G(-1, \Theta) = 1$ and the proposition applies. Second, if μ is sufficiently dispersed, then $G(1, \Theta) - G(-1, \Theta) = 1$ only if $\Theta_A - \Theta_B$ is small. For example, consider the type I preferences of example 1 when there is a single issue, $H = R^1$. Let $\Theta_A - \Theta_B = \varepsilon$ and $\frac{\Theta_A + \Theta_B}{2} = \overline{\Theta}$, let c be fixed, and assume d is distributed normally with mean 0 and variance 1. Then $G(1, \Theta) - G(-1, \Theta) = \frac{1}{\sqrt{2\pi}}\int_{\overline{\Theta} - \frac{c}{|\varepsilon|}}^{\overline{\Theta} + \frac{c}{|\varepsilon|}} e^{-\frac{x^2}{2}}\,dx < 1$ whenever $\varepsilon > 0$. For type II preferences[5] with $U(\Theta, d) = -|\Theta - d|^{\frac{1}{2}}$, $G(1, \Theta) - G(-1, \Theta) = 0$ if $|\Theta_A - \Theta_B| = |\varepsilon| \leq 4c^2$. Otherwise, $G(1, \Theta) - G(-1, \Theta) > 0$.

Another thing to notice is that if $G(1, \Theta) - G(-1, \Theta) = 1$, then expected turnout is always zero since no voter ever has preferences d and costs c which provide any gain from voting, even if all others abstained. Finally, notice that it is also true that expected turnout is zero under minimax regret behavior (from lemma 2) if $G(1, \Theta) - G(-1, \Theta) = 1$. Thus, this situation is somewhat uninteresting, except that it exactly describes the equilibrium if $\Theta_A = \Theta_B$. As we will see below, candidate competition may well produce $\Theta_A = \Theta_B$ as a final result and, therefore (by proposition 1), no turnout.

Proposition 2. Given $\Theta_A, \Theta_B, \mu)$, if $G(r, \Theta_A, \Theta_B)$ is continuous in $r \in (-\infty, \infty)$, there is a symmetric voters' equilibrium for $(\Theta_A, \Theta_B, \mu)$.

Proof. Let $h^a(\alpha, \beta) = f[q_A(\alpha), q_B(\beta)]$ and $h^B(\alpha, \beta) = f[q_B(\beta), q_A(\alpha)]$, where $q_A(\alpha) = 1 - G[\frac{1}{\alpha}, \Theta]$ and $q_B(\beta) = G(-\frac{1}{\beta}, \Theta)$. Then $q_A(\cdot), q_B(\cdot)$ are defined for $\alpha, \beta \in (0, 1)$. Let $q_A(0) = 0$ and $q_B(0) = 0$. It is then easy to show that the function $h(\alpha, \beta) = [h^a(\alpha, \beta), h^B(\alpha, \beta)]$ continuously maps $[0, 1] \times [0, 1]$ into itself, since G and f are continuous respectively in r and (α, β). By Brouwer's theorem, there is a fix-point (α^*, β^*). Let $q^*_A = q_A(\alpha^*)$ and $q^*_B = q_B(\beta^*)$. Then $(q^*_A, q^*_B, \alpha^*, \beta^*)$ is an equilibrium.

Remarks. There are a variety of easily acceptable assumptions on $U(\Theta, d)$ and μ, such that $G(r, \Theta)$ is continuous in r. For example, let $D = R^k$, $k < \infty$, and assume (1) $U(\Theta, d)$ is continuous in Θ for all $d \in D$ and (2) for each Borel subset $R \subseteq D \times (\underline{c}, \infty)$, $\mu(R) = \int_R h(e)de$, where h is a continuous density function such that $h(e) > 0$ for all $e \in E$. In fact, these conditions are stronger than necessary.

Perhaps unfortunately, if there are only a finite number of types in E (that is, E is a finite set), it is usually the case that G is not continuous in r. This does not mean there is no equilibrium; however, proposition 2 does not cover this case.

A simple corollary of proposition 2 is that if there is a positive probability that someone will vote if all others abstain, expected turnout [that is, $(n + 1)(q^*_A + q^*_B)$] is positive in equilibrium.

Corollary 2.1. Given $(\Theta_A, \Theta_B, \mu)$, such that G is continuous in r and $G(1, \Theta) - G(-1, \Theta) < 1$, then a voters equilibrium exists and $q^*_A + q^*_B > 0$.

Proof. If $q^*_A + q^*_B = 0$, then $q^*_A = q^*_B = 0$ and $\alpha^* = \beta^* = 1$. But then $q^*_A + q^*_B = 1 - G(1, \Theta) + G(-1, \Theta) > 0$ by assumption. *QED.*

C. Comparison to Minimax Regret

As an interesting side issue, one might wish to compare expected percentage turnout, $q^*_A + q^*_B$, in this expected utility model with that predicted by the minimax regret model of Ferejohn and Fiorina. The first obvious fact is that if $\alpha^*, \beta^* \leq \frac{1}{2}$, then $t_E \geq t_M$. If one is tempted to conclude from this that "since α and β are small, $t_M > t_E$," one would be wrong. To see why, consider type I preferences on a single-dimensional issues space, $U(e, \Theta) = -(a - \Theta)^2$. Let $c = 1$ (that is, normalize u by c) and assume a is normally distributed with mean 0 and variance σ. Further assume $\Theta_A = \frac{d}{2}$ and $\Theta_B = -\frac{d}{2}$. Then in voters' equilibrium $\alpha = \beta$ and $q_A(\alpha) = q_B(\alpha) = 1 - \frac{1}{\sqrt{\pi}}\int_0^{\frac{1}{\alpha d\sigma}} e^{-\frac{x^2}{2}}dx = q(\alpha d\sigma)$. Thus, implicitly, $\alpha = f[q(\alpha d\sigma), q(\alpha d\sigma)]$, or explicitly: $\alpha = h(d\sigma)$. Now $h(0) = 1$ and $h'(d\sigma) = \frac{(f_x + f_y)q'}{1-(f_x + f_y)q'} < 0$, since $q' > 0$ and[6] $f_x + f_y < 0$. Further, $\lim_{d\sigma\to\infty} h(d\sigma) \cong \frac{\sqrt{2}}{n\pi}$ for large n. Thus for small values of σd, $\alpha^* > \frac{1}{2}$, which implies $t_E > t_M$, while for large values of σd, $\alpha^* < \frac{1}{2}$, which implies $t_M > t_E$. I have not calculated the value of $d\sigma$ for which $h(\sigma d) = \frac{1}{2}$. In any case, with symmetric type I preferences, close and symmetric platforms, and small variances in tastes, higher turnout is predicted by this model than by the minimax regret model. Large variances and distant platforms lead to the opposite conclusion.

If we consider type II preferences where $U(\Theta, a) = -|\Theta - a|^{\frac{1}{2}}$ and let $\Theta_A = \frac{d}{2}$, $\Theta_B = -\frac{d}{2}$ and a be distributed normally with mean 0 and variance 1, then we find again, that for small values of d, α is near 1. For $d \leq 2c^2$, $\alpha = 1$. However, now $\frac{d\alpha}{d\sigma} > 0$ and therefore large values of σ imply $\alpha > \frac{1}{2}$. Thus, in the case of type II preferences, close and symmetric platforms and large variance in tastes lead to a higher prediction of turnout than under minimax regret behavior. That is, the effect of the variance of the tastes of voters is exactly opposite under type I and type II preferences.

In summary, given platforms, Θ_A and Θ_B, and a distribution of preferences and costs, μ, a voters' equilibrium can be defined and shown to exist if G is continuous. In general, expected turnout seems to be positive, although no precise figures were calculated. Further, whether more or less turnout is predicted by this model, as opposed to the minimax regret model, depends on the specific values of Θ_A and Θ_B, the form of preferences, and their variance. Since the choice of platforms is so crucial to that question, we turn now to modeling how they are chosen.

II. ELECTORAL EQUILIBRIUM: TWO CANDIDATES

A. Definition of Equilibrium

From section I, given $(\Theta_A, \Theta_B, \mu)$, a natural concept of voters' (partial) equilibrium arises from which one can infer, for each voter, (q_A^*, q_B^*, q_o^*): their probabilities of voting for A or B, or abstaining. Thus, given $(\Theta_A, \Theta_B, \mu)$, one can compute, assuming voters are in equilibrium, such things as (i) the probability that A wins, which is

$$\text{prob}\,\{n_A > n_B\} + \frac{1}{2}\,\text{prob}\,\{n_A = n_B\} =$$

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{r=1}^{n-2k+1} \binom{n+1}{k+r}\binom{n-k-r+1}{k} (q_A)^{k+r} (q_B)^k (1-q_A-q_B)^{n-2k-r+1}$$

$$+ \frac{1}{2} \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \binom{n+1}{k}\binom{n-k+1}{k} (q_A)^k (q_B)^k (1-q_A-q_B)^{n-2k+1}$$

or (ii) A's expected plurality, which is $(n+1)(q_A - q_B)$, or (iii) A's expected vote, which is $(n+1)\,q^A$. Each of these has been proposed, along with others,[7]

as a possible objective function for candidate A. We will begin by considering expected plurality and will reserve comment on the others until later in section IIIA.

ASSUMPTION 4. (a: Expected plurality hypothesis). Given $(\Theta_A, \Theta_B, \mu)$, both candidates act as if they wish to maximize expected plurality under the assumption that voters are in equilibrium. (That is, A desires to maximize $(n+1)\,[q_A^*(\Theta_A, \Theta_B, \mu) - q_B^*(\Theta_A, \Theta_B, \mu)]$.)

(b: Existence of voters' equilibrium). Given $(\mu, \Theta_A, \Theta_B)$, either $G(1, \Theta) - G(-1, \Theta) = 1$ or $G(r, \Theta)$ is continuous in r.

Under this assumption, there is a natural concept of electoral equilibrium.

Definition. The 4-triple $(\hat{\Theta}_A, \hat{\Theta}_B, \hat{q}_A, \hat{q}_B)$ is an *electoral equilibrium* for μ if (a) there are α and β, such that $(\hat{q}_A, \hat{q}_B, \alpha, \beta)$ is a voters' equilibrium for $(\hat{\Theta}_A, \hat{\Theta}_B, \mu)$, and (b)

$$W^A(\hat{\Theta}_A, \hat{\Theta}_B) \geq W_A(\Theta_A, \hat{\Theta}_B)\ \forall\ \Theta_A \in H$$
$$W^B(\hat{\Theta}_A, \hat{\Theta}_B) \geq W^B(\hat{\Theta}_A, \Theta_B)\ \forall\ \Theta_B \in H$$

where $W^A(\Theta_A, \Theta_B) = q_B(\Theta_A, \Theta_B, \mu) - q_A(\Theta_A, \Theta_B, \mu)$, $W^B(\Theta_A, \Theta_B) = -W^A(\Theta_A, \Theta_B)$, *and* $[q_A(\Theta_A, \Theta_B, \mu), q_B(\Theta_A, \Theta_B, \mu), \alpha, \beta]$ for some (α, β) is a voters' equilibrium for $(\Theta_A, \Theta_B, \mu)$.

Thus, $(\hat{\Theta}_A, \hat{\Theta}_B)$ is a Nash equilibrium of the (zero-sum) game in which candidates' payoffs are their expected plurality under the assumption that voters will vote as if in voters' equilibrium.

B. Existence of Equilibrium

It is easy to show that if $q_A(\Theta_A, \Theta_B, \mu)$ is concave in Θ_A and convex in Θ_B and if $q_B(\Theta_A, \Theta_B, \mu)$ is concave in Θ_B and convex in Θ_A, then an electoral equilibrium exists. However, these concavity properties need not be valid for arbitrary classes of preferences, $U(\Theta, d)$, and priors, μ. Therefore we need to explore for what preferences and priors an equilibrium *does* exist. It turns out that both the question of existence and the character of equilibrium depend crucially on the concavity properties of the utility functions and the symmetry (or lack of it) of the prior distribution. Thus, we need to consider several cases.

(i) CONCAVE UTILITY: SYMMETRIC PRIOR. We first prove that if tastes are concave in Θ and the prior, μ, is symmetric around $\hat{\Theta}$, then $\Theta_A = \Theta_B = \hat{\Theta}$ and $q_A = q_B = 0$ is an electoral equilibrium. We will then discuss the implications of weakening some of the assumptions.

Proposition 3. If

(i) $H \equiv R^k$, $k < \infty$.

(ii) (concave utility). For each $d \in D$, $U(\Theta, d)$ is concave in, and $\nabla U = [\partial U/\Theta_1, \ldots, \partial U/\partial\Theta k]$ exists for all $\Theta \in H$.

(iii) (symmetric priors). There is a $\hat{\Theta}$ such that, for all $\gamma \in R^k$, $\mu(\{e \varepsilon E | \nabla U(\hat{\Theta}, d)\cdot\gamma \leq -c\}) = \mu(\{e \varepsilon E | \nabla U(\hat{\Theta}, d)\cdot\gamma \geq c\})$, then $(\Theta, \Theta, 0, 0)$ is an electoral equilibrium.

Proof. Let $\Theta_A = \Theta_B = \hat{\Theta}$. Then, by definition of voters' equilibrium and proposition 1, $q_A(\Theta_A, \Theta_B, \mu) = q_B(\Theta_A, \Theta_B, \mu) = 0$ and $\alpha(\Theta_A, \Theta_B, \mu) = \beta(\Theta_A, \Theta_B, \mu) = 1$. We must show that there do not exist $\lambda > 0$, γ, ε, R^k, such that $1 - G(\frac{1}{\hat{\alpha}}, \hat{\Theta} + \lambda\gamma, \hat{\Theta}) - G[-\frac{1}{\hat{\beta}}, \hat{\Theta} + \lambda\gamma, \hat{\Theta}] > 0$, where $\hat{\alpha} = \alpha(\hat{\Theta} + \lambda\gamma, \hat{\Theta})$ and $\beta = \beta(\hat{\Theta} + \lambda\gamma, \hat{\Theta})$. A symmetric argument will cover B. By concavity of U in Θ, $U(\hat{\Theta} + \lambda\gamma, d) \leq U(\hat{\Theta}, d) + \lambda\nabla U \cdot \gamma$ for all $\lambda > 0$. From this, it is easy to show that $G(r, \hat{\Theta} + \lambda\gamma, \hat{\Theta}) = \mu(\{e \mid \frac{U(\hat{\Theta} + \lambda\gamma, d) - U(\hat{\Theta}, d)}{2c} \leq r\})$ $\geq \mu(\{e \mid \frac{\lambda\nabla U \cdot \gamma}{2c} \leq r\})$, since $e \in \{e \mid \lambda\nabla U\gamma \leq 2rc\}$ implies $e \in \{e \mid U(\hat{\Theta} + \lambda\gamma, d) - U(\hat{\Theta}, d) \leq 2rc\}$. Thus, if $1 - G(\frac{1}{\hat{\alpha}}, \hat{\Theta} + \lambda\gamma, \hat{\Theta}) - G(-\frac{1}{\hat{\beta}}, \hat{\Theta} + \lambda\gamma, \hat{\Theta}) > 0$, then $1 - \mu(\{e \mid \lambda\nabla U\gamma \leq \frac{2rc}{\hat{\alpha}}\}) - \mu(\{e \mid \lambda\nabla U \cdot \gamma \leq -\frac{2c}{\hat{\beta}}\}) > 0$. By condition (iii), this implies

$$1 - \mu(\{e \mid \frac{\lambda\nabla U \cdot \gamma}{2c} \leq \frac{1}{\hat{\alpha}}\}) - \mu(\{e \mid \frac{\lambda\nabla U\gamma}{2c} \geq \frac{1}{\hat{\beta}}\}) > 0. \quad (5)$$

Now from (4), $\hat{\alpha} - \hat{\beta} = f(q^A, q^B) - f(q^B, q^A) = (q^B - q^A)\,\Gamma(q^A, q^B)$, where[8] $\Gamma(\cdot) > 0$. Therefore, if $q^A(\hat{\Theta} + \lambda\gamma, \hat{\Theta}) - q^B(\hat{\Theta} + \lambda\gamma, \hat{\Theta}) > 0$, then $\hat{\alpha} < \hat{\beta}$ or $\frac{1}{\hat{\alpha}} > \frac{1}{\hat{\beta}}$. But then it is true that $\mu(\{e \mid \lambda\nabla U\gamma \leq \frac{1}{\hat{\alpha}}\}) + \mu(\{e \mid \lambda\nabla\gamma \geq \frac{1}{\hat{\beta}}\}) \geq 1$, which contradicts (5). *QED.*

Let us look at each assumption to check its severity. Condition (i) rules out, for instance, the social choice example and others where the alternative set is finite. It also implies that issues can be measured. This is unfortunate, but standard, in spatial election models. Condition (ii) is also standard in these models, natural to an economist, and allows type I preferences: $[U(\Theta, d) = -(\Theta-d)'(\Theta-d)]$. However, as we will see later, there is some question about the empirical validity of these preferences. Further, the entire character of equilibrium is altered if preferences are not concave. We will look at these issues in detail in section IIB (iii).

Given the assumption of concave preferences, condition (iii) is the crucial restriction. Let us first see what it requires. If μ comes from a continuous density on E (that is, $\mu(R) = \int_R h(e)de$), then a sufficient condition for (iii) is the existence of $\hat{\Theta}$, such that for all $d \in D$ there is $d' \in D$, such that $\nabla U(\hat{\Theta}, d) = -\nabla U(\hat{\Theta}, d')$ and $h(d,c) = h(d', c)$ for all c. For type I preferences with a shift parameter, $U(\Theta, d) = -(d-\Theta)'A(d-\Theta) + \gamma\Theta$ where (A, γ) is fixed and A is symmetric positive definite, $\nabla U = 2A(d-\Theta) + \gamma$. In this case, if d is distributed by the continuous density (h) symmetrically around $\hat{d}$ (that is, $h(d+\hat{d}) = h(\hat{d}-d)$) and independently of c, then $\hat{\Theta} = d + \frac{1}{2}A^{-1}\gamma$ satisfies (iii). Thus $\hat{\Theta}$ is the ideal point of the median voter type, $\hat{d}$. In general, condition (iii) does not seem to imply a median voter outcome because of the role of c; however, if c is independently distributed from d, then (iii) requires the existence of a median voter of type $\hat{d}$ and a platform $\hat{\Theta}$ where $\nabla U(\hat{\Theta}, \hat{d}) = 0$ ($\hat{\Theta}$ is $\hat{d}$'s ideal point).

(ii) CONCAVE UTILITY: ASYMMETRIC PRIOR. Now (iii) is clearly not a necessary condition for existence. Let us see what happens if (iii) is weakened by considering a class of examples. In particular, we return to type I preferences on a single issue, $U(\Theta, d) = -(\Theta-d)^2$. Assume c is identical and known across all voters. Let d be distributed according to the continuous density function

$$h(d) = \begin{cases} \frac{a}{a+1}e^{d} & d \leq 0 \\ \frac{a}{a+1}e^{-ad} & d \geq 0 \end{cases}$$

where $a > 0$. We will consider different values of a and note that condition (iii) of proposition 3 is satisfied if and only if $a = 1$, in which case $\hat{\Theta} = 0$. For this class of examples we can prove

Proposition 4. If $H = [-m_1, m_2]$, $m_1, m_2 > 0$ and if (Θ^*_A, Θ^*_B) is an electoral equilibrium for the above example, then

$$\Theta^*_A = \Theta^*_B = \begin{cases} 0 & \text{if } a = 1 \\ m_2 & \text{if } a < 1 \\ -m_1 & \text{if } a > 1 \end{cases}$$

Proof. If $a = 1$, proposition 3 applies. If $a > 1$, let $\Theta^*_A > -m_1$ be arbitrary and suppose Θ^*_A, Θ^*_B is an electoral equilibrium. Then $W^B(\Theta^*_A, \Theta_B) \leq 0$ for all Θ_B. Let $\Theta_B = \Theta^*_A - \varepsilon$ where $\varepsilon > 0$. $G(r, \Theta^*_A, \Theta_B) = H(\Theta^*_A - \frac{\varepsilon}{2} + \frac{rc}{\varepsilon})$ where

$$H(d) = \begin{cases} \frac{a}{a+1}e^{d} & \text{if } d < 0 \\ 1 - \frac{1}{a+1}e^{-ad} & \text{if } d > 0. \end{cases}$$

Thus for ε near zero,

$$W^B = (\frac{a}{a+1})\, exp\,[-a(\Theta^*_A - \frac{\varepsilon}{2} - \frac{c}{\beta\varepsilon}) - (\frac{1}{a+1})\, exp\,[-a(\Theta^*_A - \frac{\varepsilon}{2} + \frac{c}{\alpha\varepsilon}].$$

We will show that for some ε near zero $W^B > 0$, and therefore $\Theta_A^* > m_1$ cannot be an equilibrium. Suppose $W^B \le 0$ for all $\varepsilon > 0$. Then for all

$$\varepsilon > 0, a\, exp(\Theta_A^* - \frac{\varepsilon}{2} - \frac{c}{\beta\varepsilon}) \le exp[-a(\Theta_A^* - \frac{\varepsilon}{2} + \frac{c}{\alpha\varepsilon})]$$

or

$$\Theta_A^* - \frac{\varepsilon}{2} - \frac{c}{\beta\varepsilon} + \ln a \le -a(\Theta_A^* - \frac{\varepsilon}{2} + \frac{c}{\alpha\varepsilon}).$$

This implies

$$\Theta_A^* \le \frac{\varepsilon}{2} + \frac{c}{\varepsilon(1+a)}[\frac{1}{\beta} - \frac{a}{\alpha}] - \ln a.$$

We remind ourselves that as $\varepsilon \to 0$, $q_A \to 0$ and $q_B \to 0$. Thus as $\varepsilon \to 0$, $\alpha \to 1$ and $\beta \to 1$. Since $a > 1$, $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\frac{1}{\beta} - \frac{a}{\alpha}] = -\infty$. Thus, if $a > 0$ and $\Theta_A^* > m_1$, there is some ε near zero such that $\Theta_B \in H$ and $V^B > 0$. This establishes the proposition for $a > 1$ since a symmetric argument applies for $\Theta_B^* > -\infty$. For $a<1$ as similar proof applies. *QED*.

The key fact to note, in understanding why the proposition is true, is that if Θ_A and Θ_B are very close to each other, then, because of the type I (quadratic loss) preferences, it is only the voters in the tails of the distribution (the extreme positive and negative values of d) who will vote.[9] Thus, for example, if $a > 1$, then voters with extreme negative d are more likely [10] than voters with extreme positive d. Thus, platforms move in a negative direction. This observation extends to more general density functions and to multidimensional issue spaces.

The problem with this fact is that boundary points cannot be equilibria.

Corollary 4.1. For the class of examples covered in proposition 4, if m_1 and m_2 are large enough, an equilibrium exists if and only if $a = 1$.

Proof. (if) follows from proposition 3. (only if) Suppose $a > 1$. If (Θ_A^*, Θ_B^*) is an equilibrium, then $\Theta_B^* = -m_1$. Let $\hat{\Theta}$ be such that $\frac{a}{a+1} \in \hat{\Theta} = \frac{1}{2}$. That is, $\hat{\Theta}$ is the median voter's ideal point, $\frac{1}{2} \le \ln \frac{a+1}{2a} < 0$ for $a > 1$. Let $\Theta_A = \Theta_B + \varepsilon = -m_1 + \varepsilon$ and let $-m_1 + \frac{\varepsilon}{2} + \frac{c}{\alpha\varepsilon} = \hat{\Theta}$. Thus $\Theta_A = -\hat{\Theta} - \sqrt{(m_1 - \hat{\Theta})^2 - \frac{2c}{\alpha}}$. (By m_1 "large enough," we mean that $(m_1 - \hat{\Theta})^2 > \frac{2c}{\alpha}$.) Then $q_A = 1 - G(\frac{1}{\alpha}, \Theta) = 1 - H(\Theta_B + \frac{\varepsilon}{2} + \frac{c}{\alpha\varepsilon}) = \frac{1}{2}$. $q_B = H(\Theta_B + \frac{\varepsilon}{2} - \frac{c}{\beta\varepsilon}) < \frac{1}{2}$. Therefore, $q_A > q_B$ and $\Theta_B^* = -m_1$ cannot be an equilibrium. A similar argument follows for $0 < a < 1$. *QED*.

One must conclude that symmetric tails are a necessary condition for the existence of an equilibrium. Is this sufficient? Surprisingly, it seems so, sub-ject to a precise definition of "tails." Returning to a single issue space with $U = -(\Theta - d)^2$, remember that $V^A = q_A - q_B = 1 - H(\bar{\Theta} + \frac{C}{\alpha\varepsilon}) - H(\bar{\Theta} - \frac{c}{\beta\varepsilon})$ where $\varepsilon = \Theta_A - \Theta_B > 0$ and $\bar{\Theta} = \frac{\Theta_A + \Theta_B}{2}$. Now consider $\partial V^A/\partial\Theta_A$

$$= -h(\Theta + \frac{c}{\alpha\varepsilon})[\frac{1}{2} - \frac{c}{\alpha\varepsilon^2} - \frac{c}{\alpha^2\varepsilon}\frac{\partial\alpha}{2\Theta_A}] - h[\bar{\Theta} - \frac{c}{\beta\varepsilon}] \times [\frac{1}{2} + \frac{c}{\beta\varepsilon^2} + \frac{c}{\beta^2\varepsilon}\frac{\partial\beta}{\partial\Theta_A}].$$

For ε large enough $\partial V^A/\partial\Theta_A < 0$. In particular, if $\frac{1}{2} - \frac{c}{\alpha\varepsilon^2} - \frac{c}{\alpha^2\varepsilon}\frac{\partial\alpha}{\partial\Theta_A} > 0$, then A will want to decrease Θ_A. Similarly, B will want to increase Θ_B if $\frac{1}{2} - \frac{c}{\beta\varepsilon^2} - \frac{c}{\beta^2\varepsilon}\frac{\partial\beta}{\partial\Theta_B} > 0$. Since $\alpha = \beta \le 1$ at equilibrium, we know that there is an $\hat{\varepsilon}$ such that if $1 - H(\hat{\Theta} + \frac{c}{\varepsilon}) = H(\hat{\Theta} - \frac{d}{\varepsilon})$ for all $0 < \varepsilon \le \hat{\varepsilon}$, then an equilibrium exists at $\hat{\Theta}$. Another way of stating this is that $h(d) = h(-d)$ for all $d \ge \hat{\Theta} + \frac{c}{\hat{\varepsilon}}$.

It should be noted that the more concave the $U(\cdot)$ are, the larger the tail which must be symmetric. Thus, for example, if $U(\Theta, d) = -(\|\Theta - d\|)^v$ for $v \ge 1$ where $\|x\| = (\Sigma x_i^2)^{1/2}$, then larger v require larger $\hat{\varepsilon}$. In the best case ($v = 1$) with a single issue, $\hat{\varepsilon} = 0$; that is, no symmetry is required for existence at the median. In this case, for $A > B$, $q_A = \mu(\{e \mid \frac{A-B}{2c} \ge \frac{1}{\alpha}$ and $d \ge \frac{A+B}{2} + \frac{1}{\alpha}\})$ and $q_B = \mu(\{e \mid \frac{A-B}{2c} \ge \frac{1}{\beta}$ and $d \le \frac{A+B}{2} - \frac{1}{\beta}\})$. If $A = \hat{\Theta}$ where $\mu(\{e \mid d \le \hat{\Theta}\}) = \frac{1}{2}$, then, for all $B < A$, either $q_A = q_B = 0$ (when $\frac{A-B}{2c} < 1$) or $q_A \ge \frac{1}{2}$ and $q_B \le \frac{1}{2}$ and, therefore, $V^A(\Theta_A, \Theta_B) \ge 0$ for all $\Theta_B \ne \Theta_A = \hat{\Theta}$. The conclusion one draws from all of this is that if preferences are concave, a sufficient amount of symmetry of tastes must occur if an equilibrium is to exist. This lack of robustness of the model is somewhat discouraging.

However, one must recognize that type I preferences, as well as strictly concave utility functions, imply a form of behavior which seems to be empirically invalid. I refer, in particular, to a stylized empirical fact: abstentions increase with alienation. That is, as both candidates' platforms move away from a voter's ideal platform, the voter is more likely to abstain. With concave and type I preferences, however, just the opposite is predicted. This is easiest to see by writing $\Delta = U(\Theta_A, d) - U(\Theta_B, d) = -(-\Theta_A + d)^2 + (-\Theta_B + d)^2 = 2(d\varepsilon - \overline{\Theta}\varepsilon)$, where $\varepsilon = \Theta_A - \Theta_B$ and $\overline{\Theta} = \frac{\Theta_A + \Theta_B}{2}$. Now $\frac{\partial \Delta}{\partial \overline{\Theta}} = -2\,\varepsilon\,\overline{\Theta}$. If $\varepsilon > 0$, $\frac{\partial \Delta}{\partial \overline{\Theta}} < 0$ and, therefore, as $\overline{\Theta}$ declines, this voter is more likely to vote for A. From our model (assuming α and β constant for now), a voter abstains if $-\frac{1}{\beta} < \frac{\Delta}{2c} < \frac{1}{\alpha}$ or $-\frac{1}{\beta} < (d - \overline{\Theta})\,\varepsilon < \frac{1}{\alpha}$. Thus abstentions occur for small values of $|\overline{\Theta} - d|$ and not for large values. This is contrary to the stylized fact. We thus turn to a consideration of other types of preferences.

(iii) NONCONCAVE UTILITY: EXAMPLES. As an alternative to type I preferences we consider those of type II, or $U(\Theta, d) = -\|\Theta - d\|^{1/2}$. These are convex functions on each side of the ideal point d, even though they are not convex over all Θ. Assume $\Theta_A > \Theta_B$ and let $r = \frac{2c}{\alpha}$ and $s = \frac{2c}{\beta}$. If $H = R^1$ (a single issue), then a voter with parameters (d, c)

(i) votes for A if $\Theta_A - \Theta_B > r^2$ and $\Theta_A - z(r) \le d \le \Theta_A + x(r)$

(ii) votes for B if $\Theta_A - \Theta_B > s^2$ and $\Theta_B - x(s) \le d \le \Theta_B + z(s)$

and (iii) abstains otherwise,

where

$$z(w) = \frac{\Theta_A - \Theta_B}{2} - w\left(\frac{\Theta_A - \Theta_B}{2} - \frac{w^2}{4}\right)^{1/2}$$

and

$$x(w) = \left(\frac{\Theta_A - \Theta_B}{2} - \frac{w^2}{2}\right)^2 w^{-2}.$$

Notice that if $\Theta_A - \Theta_B$ is fixed and if Θ_A and Θ_B simultaneously move far enough away from d, the voter will abstain. Let $H(\cdot)$ be the distribution function of d (and $h(\cdot)$ the density function) and assume c is fixed and identical for all voters. Then the probability that a voter votes for A is

$$q^A = \begin{cases} h(\Theta_A + x(r)) - H(\Theta_A - z(r)) & \text{if } \Theta_A - \Theta_B > r^2 \\ 0 & \text{otherwise} \end{cases}$$

and the probability that a voter votes for B is

$$q^B = \begin{cases} H(\Theta_B - x(s)) - H(\Theta_B + z(s)) & \text{if } \Theta_A - \Theta_B > r^2 \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition can be established for this class of examples.

Proposition 5. If $H(\cdot)$ is continuous and unimodal at Θ_m (i.e., $h'(d) \ge 0$ for $d \le \Theta_m$ and $h'(d) \le 0$ for $d \ge \Theta_m$) and if Θ_A^*, Θ_B^* is an equilibrium for h, then $\Theta_m - 4c^2 \le \Theta_A^*, \Theta_B^* \le 4c^2 + \Theta_m$.

Proof. Suppose $\Theta_B^* < \Theta_m - 4c^2$. Then there is $\varepsilon > 0$ such that $\Theta_A = \Theta_B + 4c^2 + \varepsilon < \Theta_m$. Now $\varepsilon \to 0$ implies $x \to 0$ and $z \to 0$, since $\varepsilon \to 0$ implies $r \to 2c$ and $s \to 2c$ (because $\alpha \to 1$ and $\beta \to 1$). [Note that for $\varepsilon > 0$, $\Theta_A - \Theta_B > r^2$ and $\Theta_A - \Theta_B > s^2$, since otherwise $q_A = q_B = 0$, which implies $\alpha = \beta = 1$, which

implies $\Theta_A - \Theta_B > r^2, s^2$.] Assume $s = r$. There exists $\varepsilon > 0$ such that $\Theta_A(\varepsilon)$ $+ x(\varepsilon) < \Theta_m$. Therefore $h(t) \ge h[\Theta_A(\varepsilon) - z(\varepsilon)]$ for all $t\,\varepsilon[\Theta_A - z, \Theta_A + x]$ and, since $\Theta_B + z \le \Theta_A - z$, $h(t) \le h(\Theta_B + z) \le h(\Theta_A - z)$ for all $t\,\varepsilon[\Theta_B -$ $x, \Theta_B + z]$. Thus $q_B(\Theta_A(\varepsilon), \Theta_B^*) < q_A(\Theta_A(\varepsilon), \Theta_B^*)$ if $r = s$. Suppose $r \ne s$ and $q_B > q_A$. At a voters' equilibrium[11] if $q_B > q_A$, then $s > r$ since $\beta < \alpha$. Now $\partial x/\partial s = -\frac{1}{2s^3}\left(\frac{\Theta_A - \Theta_B}{2} - \frac{s^2}{2}\right)^2 - \frac{1}{s}\left(\frac{\Theta_A - \Theta_B}{2} - \frac{s^2}{2}\right) < 0$ and $\partial z/\partial s = -\left(\frac{\Theta_A - \Theta_B}{2} - \frac{s^2}{4}\right)^{1/2} + \left(\frac{s}{2}\right)^2\left(\frac{\Theta_A - \Theta_B}{2} - \frac{s^2}{4}\right)^{-1/2} < 0$ since $-\left[\frac{\Theta_A - \Theta_B}{2} - \frac{s^2}{4}\right] + \frac{s^2}{4} = -\left(\frac{\Theta_A - \Theta_B}{2} - \frac{s^2}{2}\right) < 0.$

Thus $\partial q_B/\partial s = h(\Theta_B + z)\cdot\partial z/\partial s + h(\Theta_B - x)\,\partial x/\partial s < 0$. Therefore $q_B(h) < q_B(r) < q_A(r)$ implies that if $\Theta_B^* < -4c^2 + \Theta_m$, there is $\Theta_A < \Theta_m$ such that $q_A > q_B$. Thus Θ_B^* cannot be an equilibrium platform. A similar proof works for $\Theta_A > 4c^2 + \Theta_m$ and symmetry implies the rest of the proposition. *QED.*

We have shown that for a unimodal distribution of tastes and $U = -|\Theta - d|^{1/2}$, any equilibrium must be concentrated around the *mode*.[12] Since this equals the median only for symmetric distributions, we immediately see that non-concavities produce qualitatively different equilibria.

Although proposition 5 contains necessary conditions for equilibrium platforms, they are not sufficient. In fact, if the mean and the mode are too far apart (relative to $4c^2$) or—what is the same thing—if the distribution of tastes is too skewed, there may be no equilibrium. Consider the following continuous, asymmetric density function for d.

$$h(d) = \begin{cases} S\cdot e^{bd} & \text{if } d \le 0 \\ (1 - \varepsilon d)S & \text{if } 0 \le d \le R \\ e^{-a(d-R)}S(1 - \varepsilon R) & \text{if } d \ge R \end{cases}$$

where $S = [\frac{1}{b} + R(1 - \frac{\varepsilon}{2}R) + \frac{1}{a}(1 - \varepsilon R)]^{-1}$, $b, \varepsilon, a, R > 0$, and $1 - \varepsilon R > 0$. Note that h is unimodal, where $d = 0$ is the mode. Let $\Theta_A = \Theta_B + \eta$ and assume that $r = s$. Then $x = \frac{1}{r^2}(\frac{\eta}{2} - \frac{r^2}{2})^2$. For (η, R) such that $x > 4c^2$, and $R \ge x$, if $\Theta_B \le \Theta_m + 4c^2 = 4c^2$, then

$q^A - q^B =$

$S\cdot[(\Theta_A + x)(1 - \frac{\varepsilon}{2}(\Theta_A + x)) - (\Theta_A - z)(1 - \frac{\varepsilon}{2}(\Theta_A - z)) -$

$((\Theta_B + z)(1 - \frac{\varepsilon}{2}(\Theta_B + z)) + \frac{1}{b}) - \frac{1}{b}e^{+b(\Theta_B - x)}] = \frac{1}{b}S(1 + e^{+b(\Theta_B - 2)}) +$

$S[x - \Theta_B] - \frac{\varepsilon}{2}S[(\Theta_A + x)^2 - (\Theta_A - z)^2 - (\Theta_B + z)^2] > 0,$

if $x - \Theta_B > \frac{\varepsilon}{2}[(\Theta_A + x)^2 - (\Theta_A - z)^2 - (\Theta_B + z)^2]$.

Since $x - \Theta_B > 0$, we can choose $\varepsilon > 0$ as well as η and R, such that, for all $\Theta_B \leq 4c^2$, $q_A - q_B > 0$. Therefore, from proposition 5, if ε is small enough and R is large enough, there can be no equilibria, since there are enough voters to the right of B (relative to those to the left of the median) to enable A to always collect a majority of those who vote.

One can state (overly strong) sufficient conditions for the existence of an equilibrium in this class of examples.

Proposition 6. If $h(d)$ is continuous and unimodal, $U = -|\Theta - d|^{1/2}$, d, $\Theta \in R'$, and c is fixed, let $\hat{\Theta}$ maximize $H(\Theta_A + 4c^2) - H(\Theta_A - 4c^2)$. Let $P(\hat{\Theta}) = H(\hat{\Theta} + 4c^2) - H(\hat{\Theta} - 4c^2)$. If $P(\hat{\Theta}) \geq H\hat{\Theta} - 4c^2)$ and $P(\hat{\Theta}) \geq 1 - H(\hat{\Theta} + 4c^2)$, then $\Theta_A^* = \Theta_B^* = \hat{\Theta}$ is an equilibrium.

Proof. Let $\Theta_A = \hat{\Theta}$ and $\Theta_B < \Theta_A$. If $\Theta_B \geq \Theta_A - 4c^2$, then $q_A = q_B = 0$. If $\Theta_B < \Theta_A - 4c^2$, one can show $\Theta_B + z \leq \Theta_A - 4c^2 < \Theta_m$. If $x \leq 4c^2$, then $h(r) \geq h(\Theta_B + z)$ for all $v = \varepsilon [\Theta_A - z, \Theta_A + x]$ and $h(t) \leq h(\Theta_B + z)$ for all $t \in [\Theta_B - x, \Theta_B + z]$. Therefore $q_A > q_B$. If $x \geq 4c^2$, $q_A - q_B = H(\Theta_A + x) - H(\Theta_A - z) - H(\Theta_B + z) + H(\Theta_B - x) \geq H(\Theta_A + 4c^2) - H(\Theta_a - 4c^2) + H(\Theta_B - x) - H(\Theta_A - z)$. By assumption, $P(\Theta_A) \geq H(\Theta_A - 4c^2) \geq H(\Theta_A - z)$. Thus, $q_A > q_B$. A similar argument follows for $\Theta_B = \hat{\Theta}$ and $\Theta_A > \Theta_B$. *QED.*

One is led naturally to the following

Conjecture. If $U = -\|\Theta - d\|^{1/n}$ for $n > 1$, $\Theta, d \in R^k$, $\|x\| = (\Sigma x_k^2)^{1/2}$ and if the density on (d, c) is $h(d)g(c)$ where h is continuous and unimodal [i.e., $\exists\, \Theta_m \in \nabla h(d) \cdot (\Theta_m - d) > 0 \; \forall\, d$] and, letting $\bar{p} = \max_{\Theta} \mu(\{(d,c) \mid \frac{\|\Theta - d\|^{1/n}}{2c} \leq 1\})$ and $\hat{\Theta}$ be the solution, if $\hat{p} \geq \mu(\{(d,c) \mid \alpha(d - \hat{\Theta}) \geq 0$ and $\frac{\|\hat{\Theta} - d\|^{1/n}}{2c} > 1\})$ for all $\gamma \neq 0$ then $\Theta^A = \Theta^B = \hat{\Theta}$ is an equilibrium.

Thus, if this conjecture is correct, then when preferences are of the form $-\|\Theta - d\|^{1/n}$, $(n > 1)$, and the distribution of tastes is unimodal and not too asymmetric, equilibrium exists where $\Theta_A = \Theta_B$, $q_A = q_B = 0$, and Θ_A and Θ_B are somewhere near the mode.

C. Summary and Comments

We can summarize the results of this section in several brief statements.

1. If utility functions are concave and tastes are continuously and symmetrically distributed, then an electoral equilibrium exists, with both candidates selecting the *median* voter's ideal point, and no one votes.

2. If utility functions are concave and tastes are asymmetrically distributed in the tails, then an electoral equilibrium will not exist.

3. If preferences are of type II (convex on each side of an ideal point) and if tastes are continuously, unimodally, and not too asymmetrically distributed, then an electoral equilibrium exists, with both candidates choosing the *modal* voter's ideal point, and no one votes.

Several comments about these results seem to be in order. First, if we were to substitute minimax regret behavior for our fully rational expected utility model of voters, nothing substantive with respect to the existence of electoral equilibrium or modality of candidates platforms would be altered. None of the arguments in this section would be affected if we let $\alpha = \beta = \frac{1}{2}$. Thus *both models of behavior produce identical outcomes in equilibrium.*[15] They are, therefore, significantly different in their predictions of disequilibrium phenomena only if candidates maximize expected plurality.

Second, the role of the assumption of candidate irrelevance should not be ignored. If different voters have different beliefs about the likelihood of candidates' postelection positions, then, even if candidates were driven (by expected plurality maximization) to choose identical positions, there may be a positive probability of turnout. Further, it is highly likely, if the candidates' names (reputations) count, that equilibria with differentiated platforms and positive turnout will exist.

Third, we have not considered the implications for existence and characterization of equilibria of the assumption that preferences might be a mixture of type I and type II. It would be interesting to know, for example, the outcome (equilibrium) predicted by this model when "Republicans" have type I preferences and are concentrated to the "right" of the median voter and "Democrats" have type II preferences and are concentrated to the "left" of the median voter. Technically, one could consider a distribution of preferences constructed as a convex combination of type I and type II. For example, let $h^I(d)$ be a density of type I preferences and $h^{II}(d)$ be a density of type II preferences, and consider the implications if the actual density of tastes is $\lambda\, h^I(d) + (1 - \lambda)\, h^{II}(d)$ for some $\lambda \in [0, 1]$. We leave this exercise to the interested reader.

Fourth, we have not explored the implications of this model for the standard social choice problem with a finite set of alternatives. A reasonable approach to that problem would be to imbed that set of alternatives in the real line, extend the preferences of voters over the line, and then apply the results of this section. This is in the spirit of single-peak preferences (a property that both type I and type II preferences have on the line—although not if the issue space is multidimensional) and, of course, the method of imbedding is crucial.[14]

Finally, although I have used the phrase "general equilibrium" in the title, I have ignored at least one set of important actors and one type of candidate decision. The missing actors are political activists who donate funds (to change voters' likelihood beliefs?) and who, by ringing doorbells, can raise the cost of not voting and thereby raise turnout. The missing decision is the issue of whether to run or not. Entry into electoral competition has been ignored.

We turn next to some generalizations of the model and to the implications which arise when candidates adopt behavior other than expected plurality maximization.

III. EXTENSIONS AND ALTERATIONS

A. Other-Candidate Objective Functions

In section IIA we indicated that candidates might wish, for example, to maximize the probability of winning or their expected vote. We now consider each of these in turn.

Let us consider a scoring function where $s^i = 1$ if voter i votes for A, 0 if they abstain, and -1 if they vote for B. Then the probability that A wins is simply $P_A = \text{prob}\{\sum_{i=1}^{n} s^i > 0\} + \frac{1}{2}\text{prob}\{\sum_i s^i = 0\}$, or $\text{prob}\,\frac{1}{n}\sum_{i=1}^{n} s^i > 0\} + \frac{1}{2}\text{prob}\{\frac{1}{n} s^i = 0\}$, where $q_A = \text{prob}\{s^i = 1\}$ and $q_B = \text{prob}\{s^i = -1\}$. Since each voter is independently and identically distributed,[15] as $n \to \infty$ prob $\{|\frac{1}{n}\sum_i s^i - (q_A - q_B)| > \varepsilon\} \to 0$ for all $\varepsilon > 0$. Thus for large electorates, a reasonable approximation of the maximization of P_A is the maximization of expected plurality.[16]

One expects, therefore, that if candidates maximize their probability of winning, the outcome in large electorates will be the same as that which occurs if they maximize expected plurality.

If, on the other hand, candidates maximize expected votes, the outcomes are significantly different because of the role of abstentions. For example, there is no tendency for platforms to converge; in fact, candidates will constantly try to differentiate themselves from their opponents.

Lemma 5. If there is an $\varepsilon \geq 0$ such that $G(1, \hat{\Theta}_A, \hat{\Theta}_B) - G(-1, \hat{\Theta}_A, \hat{\Theta}_B) > 0$ whenever $\|\Theta_A - \Theta_B\| > \varepsilon$, and if Θ_A^*, Θ_B^* is an electoral equilibrium under vote maximization, then $\Theta_A^* \neq \Theta_B^*$.

Proof. If $\Theta_A^* = \Theta_B^*$, then $q_A^* = q_B^* = 0$. But either A or B can change Θ_j such that $q_j^* > 0$. *QED.*

I have not characterized further (much less established the existence of) electoral equilibrium under vote maximization. It may, however, be informative to consider an example, and so we return to a single-dimensional issue space with type I preferences and c fixed and known across all voters. As before, if $\Theta_A > \Theta_B$, $\overline{\Theta} = \frac{\Theta_A + \Theta_B}{2}$ and $\varepsilon = \Theta_A - \Theta_B$, then $G(r, \Theta) = H(\overline{\Theta} + \frac{rc}{\varepsilon})$ where H is the distribution function of ideal points, d. We consider only equilibria for which $q_A = q_B$ (whether there may be others is an open question). Under the appropriate differentiability and symmetry conditions on H, a necessary condition[17] at equilibrium is that $\frac{\partial q_A}{\partial \Theta_A} = 0$ and $\frac{\partial q_B}{\partial \Theta_B} = 0$. Thus, $\frac{1}{2} = \frac{c}{\alpha\,\varepsilon^2} = \frac{c}{\beta\,\varepsilon^2}$ and $q_A = 1 - H(\overline{\Theta} + \frac{\varepsilon}{2}) = H(\overline{\Theta} - \frac{\varepsilon}{2})$. If H is symmetric around $\hat{\Theta}$, then $\Theta_A = \hat{\Theta} + \frac{1}{2}\sqrt{\frac{2c}{\alpha}}$ and $\Theta_B = \hat{\Theta} - \frac{1}{2}\sqrt{\frac{2c}{\alpha}}$ where $\alpha = f(q_A, q_B)$. [For minimax regret behavior $\alpha = \frac{1}{2}$ and $\Theta_A = \hat{\Theta} + \sqrt{c}$, $\Theta_B = \hat{\Theta} - \sqrt{c}$.] Although α cannot be easily solved for, we know (since $q_A = q_B$) that $\frac{n!}{(n/2)!(n/2)!}(\frac{1}{2})^n \leq \alpha \leq 1$. By Stirlings formula, for large n, $\sqrt{\frac{2}{n\pi}} \leq \alpha \leq 1$. Therefore, at a vote maximizing electoral equilibrium, $(\sqrt{2c})(\frac{n\pi}{2})^{1/4} \geq \Theta_A - \Theta_B \geq \sqrt{2c}$. These are not very tight bounds, but whatever $\Theta_A - \Theta_B$ is, there is always positive turnout in this type of equilibrium.

It should be emphasized that, even for the example, only necessary conditions have been examined. I have not yet found additional conditions which guarantee that these are sufficient. Thus it is possible that an electoral equilibrium, with vote maximizing behavior on the part of the candidates, does not exist.

B. More Candidates and Non-identical Beliefs

Rather than proceed through a variety of special cases, we turn to a description of the general model of voter behavior under the extension of rationality that we have proposed. To do so, we must introduce some new notation and recall some old:

$i = 1, \ldots, n + 1$	voters	$(0 < n < \infty)$,
$j = 1, \ldots, m$	candidates	$(2 \leq m \leq \infty)$,
$\Theta_j \,\varepsilon\, T$	j's platform	
n_j	the number of votes for j,	
δ^i	i's decision function where $\delta^i = (\delta^i_o, \ldots, \delta^i_m)$ and if i votes for j ($j = 0$ is abstention), then $\delta^i_j = 1$ and $\delta^i_k = 0$ for $k \neq j$. We will let $\delta^i(j) = (0, \ldots, 0, 1, 0, \ldots, 0)$ be i's decision to vote for j,	

$\Omega_n = \{\underline{n} \,\varepsilon\, R^{m+1} \mid n_j$ is a non-negative integer and $\sum_{j=0}^{m} n_j = n\}$.
Ω_n represents all possible election outcomes in terms of votes if there are n voters,
$\Theta = (\Theta_1, \ldots, \Theta_m) \,\varepsilon\, T^m$,
$h: T^m \times \Omega \to M(T)$ where $M(T)$ is the space of probability measures on T and h is the outcome rule specifying the probability of the winning platform, if candidates have platforms $(\Theta_1, \ldots, \Theta_m)$ and voters vote $(n_o, \ldots, n_m)$.

Remark. In section II, where $m = 2$,

$$h(\Theta, \underline{n}) = \begin{cases} \Theta_A \text{ if } n_A > n_B \\ \Theta_B \text{ if } n_B > n_A \\ \Theta_j \text{ with probability } \frac{1}{2} \text{ if } n_A = n_B. \end{cases}$$

Using the above, we see that $V(\Theta, n, d^i) = \int_H U(\gamma, d^i)dh(\Theta, n)$ is i's expected utility if platforms are Θ and votes are n.

A voting strategy for voter i is a mapping from voter characteristics (d^i, c^i) to decisions. That is, $\delta^i: E \to \{\delta^i(o), \delta^i(1), \ldots, \delta^i(m)\}$. We let $Z^i = E^1 x \ldots x E^{i-1} x E^{i+1} x \ldots x E^n$ be the space of others' characteristics and represent i's beliefs about $z^i \varepsilon Z^i$ by a mapping $\Psi^i: E^i \to M(Z^i)$. Thus if i has characteristic e^i, he believes that others' are distributed according to $\Psi^i(e^i)$. With these beliefs and with knowledge of the strategies of others, $\delta^{)i(} = (\delta^1, \ldots, \delta^{i-1}, \ldots, \delta^n)$, i's expected utility from voting for j (the decision $\delta^i(j)$) is $W^i(\delta^i(j), \delta^{)i(}, e^i) \equiv \int V(\Theta, \delta^i(j) + \Sigma_{h \neq i} \delta^h(e^h), e^i) d\Psi^i(e^i) - (1 - \delta^i_o(j))C^i$.

The integral is i's expected utility of the outcome of the election and $(1 - \delta^i_o(j)) = C^i$ is the cost of his decision. That is, $(1 - \delta^i_o(j))C^i = C^i$ if $j = 1, \ldots, m$ and it is 0 if i abstains by choosing $j = 0$.

We can now state precisely the generalization of a voter equilibrium introduced in section II.

Definition. A voters' equilibrium for $<(\Theta_1, \ldots, \Theta_m), \Psi^1, \ldots, \Psi^{n+1}>$ is an $n + 1$ – triple of strategies $(\delta^1, \ldots, \delta^{n+1})$ such that for all $i = 1, \ldots, n + 1$ and all $e^i \varepsilon E^i$, $\delta^i(e^i)$ solves

$$\underset{j=0,\ldots,m.}{\text{Maximize}}\ W^i(\delta^i(j), \delta^{)i(}, e^i)$$

This concept of equilibrium is identical to that of a Bayes equilibrium.

For a variety of reasons, this model is extremely difficult to analyze without further assumptions on the structure of beliefs. Thus we introduce the following.

Assumption. (Independent–identical beliefs). For each voter i, $\Psi^i(e^i) = \mu x \ldots x \mu$ where $E^h \equiv E$ for all h and $\mu \varepsilon M(E)$.

Under this assumption, there is one voters' equilibrium which is of particular interest: the one in which all strategies are identical.

Definition. A symmetric voters' equilibrium is a voters' equilibrium such that $\delta^i(\cdot) = \delta^*(\cdot)$ for all $i = 1, \ldots, n + 1$.

We consider only symmetric equilibria throughout the rest of this paper and thus need only look at a common strategy, δ.

Given a strategy δ, the probability that a voter votes for candidate j is

$$q_j = \mu(\{e \mid \delta(e) = \delta(j)\}).$$

Thus, if all h use δ, the probability that the votes tally to $\underline{n} = (n_0, \ldots, n_m)$ when i is not considered can be computed to be

$$P(\underline{n}, q) = \frac{n!}{n_o! \ldots n_m!} (q_o)^{n_o} \ldots (q_m)^{n_m}. \quad (6)$$

Further, the probability of $\underline{n}$ if i votes for j is $P(\underline{n} - \delta^i(j), q)$. Therefore i's expected utility of voting for j given δ is

$$W_j(\Theta, e^i) = \Sigma_{n \varepsilon \Omega_{n+1}} V(\Theta, n, e^i) P(\underline{n} - \delta(j)) - (1 - \delta_o(j))C^i.$$

If we let $R_j(\Theta, p) = \{e \mid W_j(\Theta, e) > W_k(\Theta, e)$ for all $k = 0, \ldots, m$ and $k \neq j\}$, then the probability that a voter votes for j is (7) $q_j = \mu[R_j(\Theta, p)]$ for $j = 0, \ldots, m$.

Remark. δ^* is a symmetric voters' equilibrium for $<\Theta, \mu>$ if and only if (i) $\delta^*(e) = \delta(j)$ when $e \varepsilon R_j(\Theta, p^*(\cdot, q^*))$, and (ii) p^* and q^* simultaneously satisfy (6) and (7).

Remark. In applications, one need only calculate $p(n)$ for $n \ni h(\Theta, n) \neq h(\Theta, n - \delta^i(j))$ for some $j = 0, \ldots, m$. Further, one can "lump together" all $n, n' \ni h(\Theta, n) = h(\Theta, n')$ *and* $h(\Theta, n - \delta^i(j)) = h(\Theta, n' - \delta^i(j))$ for all j. In section 1, $\alpha = p_3 + p_4$ and $\beta = p_2 + p_3$ did just that (see McKelvey and Ordeshook, 1972).

With one more definition, the continuity of μ, we can state an existence result for a symmetric voters equilibrium and some implications for turnout.

Definition. $\mu \varepsilon M(E)$ is "continuous" if $A^q \to A^o \Rightarrow \mu(A^q) \to \mu(A^o)$ where $A^q \to A^o$ *iff* (1) $a^q \varepsilon A^q, a^q \to a^o \Rightarrow a^o \varepsilon A^o$ and (2) $a^o \varepsilon A_o \Rightarrow \exists\, a^q \varepsilon A^q \ni a^q \to a^o$.

Proposition 7. A symmetric voters' equilibrium exists for $<\Theta, \mu>$ if (a) μ is continuous and (b) V is continuous in e^i.

Proof. Since $q \varepsilon [0,1]^{m+1}$ and $p \varepsilon [0,1]^{n!}$, if $\mu(R_j(\Theta, p))$ is continuous in p, then, since $p(n, q)$ is continuous in q, Brouwer's theorem applies and we are done. Thus it is sufficient to note that $R_j(\Theta, p)$ is a continuous correspondence in p since V is continuous in e and W_j is linear in p. *QED.*

Remark. Remember, $V(\Theta, n, e^i) = \int_H U(\gamma, d^i)\, dh(\Theta, n)$. Therefore, if U is continuous in d, so is V. Also, if $E \times [\delta, \infty) \leq R^k$ and $\mu(D) = \int_D h(e,c)\, dcdc$ where h is continuous, then μ is "continuous."

Corollary 7.1. If the outcome function h has the property that $h(<n+1,0, \ldots, 0>, \Theta) = \Theta_j$ with probability $\frac{1}{m}\ \forall j$ and $h(<n,0, \ldots, 0> + \delta(j), \Theta) = \Theta_j$, then expected turnout is zero in a symmetric voters equilibrium if and only if $\mu(\{e \mid \frac{1}{m} \Sigma_{j=1}^m U(\Theta_j, d) \geq \max_j U(\Theta_j, d) - c\}) = 1$.

Corollary 7.2. (a) If $\Theta_1 = \ldots = \Theta_m$, then $q^*_o = 1$. (b) Let $\hat{U}(d) = \max_j U(\Theta_j, d)$ and $\underline{U}(e) = \min_j U(\Theta_j, d)$. Then (b.1) if $\mu(\{e \mid c > \frac{m-1}{m} (\hat{U}(d) - \underline{U}(d))\}) = 1$, then $q^*_o = 1$. (b.2) if $\mu(\{e \mid c < \frac{1}{m} (\hat{U}(d) - \underline{U}(d))\}) > 0$, then $q^*_o < 1$.

For type I preferences, if μ is represented by a continuous positive density on R^k and if $\Theta_j \neq \Theta_k$ for some $j, k = 1, \ldots, m$, then (b.2) obtains and expected turnout is positive.

One should not rush from corollary 7.2 to the conclusion that turnout increases as the number of candidates increases. To examine that issue, one must also consider candidate competition, given voters' behavior. Unfortunately, as the reader probably knows, M candidate competition ($M \geq 3$) is much more complex than what was analyzed in section II. Two new considerations enter. First, it is now conceivable, and likely, that voters may not vote for their most preferred candidate.[18] That is, candidate j may receive votes from voters who prefer candidate ℓ to j to k, if those voters view the probability of affecting the election of ℓ as much smaller than that of affecting the election of j. Second, if all m platforms are identical, any one candidate, j, need only ensure that $q^j > \frac{1}{m-1} q^i$, where q^i is the probability i votes for the others, to be better off. Thus as m increases, it is more probable that candidates can easily gain by shifting away from common platforms. However, if j does this and then k moves between Θ^j and Θ^i (the platform of others), k may capture most of the votes for Θ^i by the fact that they vote for their second highest alternative, and thus $\Theta^j \neq \Theta^i$ is not an equilibrium. Under certainty, equilibria with $m \geq 3$ are rare. For the model in this paper, they are more likely to exist because of the uncertainty and the possibility of abstentions; however, it is probable that equilibria with $m > 2$ are less likely than those with $m = 2$. This remains an open question.

C. Some Thoughts on Testing

One issue which is constantly raised concerns the empirical validity of a model: "Is it consistent with facts?"

Let us first consider some facts which cannot be addressed. Since $\langle \Theta, \mu \rangle$ are the only exogenous variables in the voters' model and μ is the only variable in the full model, such a question as "How is turnout affected by perceived closeness and/or party differences?" cannot be addressed, since turnout and closeness are simultaneously determined in the voters' model while turnout, closeness, and party difference are simultaneously determined in the full model. Thus regressions of the form used by Ferejohn and Fiorina (1975) are incorrectly specified in the context of this model. I must admit, it is possible that partial effects may be identifiable from some reduced form regressions; however, I suspect not. To see why, let us consider a variation on the two-candidate model in which we let a voter's characteristic be (e^i, b^i), where e^i parameterizes tastes and costs and b^i parameterizes beliefs. A strategy is now a function of (e, b). If we assume that when i is (e^i, b^i) he acts as if all other voters believe $\mu \in M(E)$ is $\mu(\cdot, b^i)$, we can partition $E \times B$ into four sets by choosing the values of two parameters (p, γ), as follows:

$$A_{11} = \{(e, b) \mid |U(\Theta_A, d) - U(\Theta_B, d)| \leq p, \alpha^* \leq \gamma\}$$
$$A_{12} = \{(e, b) \mid |U(\Theta_A, d) - U(\Theta_B, d)| \leq p, \alpha^* \geq \gamma\}$$

and so forth, where α^* is evaluated at the voters' equilibrium for b.

Thus, for example, A_{21} represents the tastes and beliefs which would yield a response that party differences are large, and the election will be close. $\mu(A_{ij})$ would be the probability that a randomly selected voter belongs in A_{ij}. Thus, by suitable choice of μ, p, and γ (and the class of preferences and beliefs, $E \times B$), one might be able to "explain" all response patterns.

This is an uncomfortable conclusion in that it seems to say the model has no predictive value with respect to voting behavior. I think, however, that is the wrong conclusion. In fact, the model predicts (in two-candidate elections) very precise outcomes. For example, if tastes are symmetrically and unimodally distributed, then both candidates' final platforms should be near the median and modal voters' choice and turnout should be light (or zero). Also, as we saw, asymmetrics and the composition of tastes (type I or type II preferences) significantly affected the predicted outcomes. Thus, in fact, the model is "testable." Further, since—in general equilibrium—both our model of voter behavior and the Ferejohn-Fiorina model predict identical outcomes, one must either await further refinements of each or use "disequilibrium phenomena" to differentiate between the two.

Needless to say, there is much more work to be done before we fully understand the complete implications of all the simultaneous interactions between voters and candidates.

APPENDIX

In this appendix we collect some results that deal with the comparative statics of a symmetric voters equilibrium for two-candidate elections.

Lemma A.1. If $G(r, \Theta_A, \Theta_B)$ has continuous second derivatives in (r, Θ) in a neighborhood of $(\frac{1}{\alpha^*}, \Theta^*)$ and $(-\frac{1}{\beta^*}, \Theta^*)$, where $(\alpha^*, \beta^*, \Theta^*)$ is a symmetric equilibrium for $\langle \Theta^*, \mu \rangle$, then the solutions, $\langle \alpha(\Theta), \beta(\Theta) \rangle$ of

$$f[q^A(\Theta, \alpha), q^B(\Theta, \beta)] - \alpha = 0$$
$$f[q^B(\Theta, \beta), q^A(\Theta, \alpha)] - \beta = 0$$

have continuous first derivatives in Θ in those neighborhoods. if

$$D = \begin{vmatrix} f_1^1 q_\alpha^A - 1 & f_2^1 q_\beta^B \\ f_2^2 q_\alpha^A & f_1^2 q_\beta^B - 1 \end{vmatrix} \neq 0$$

where f_2^1 is $\frac{\partial f(q^A, q^B)}{\partial q^B}$

Proof. Implicit function theorem.

Lemma A.2. Under the conditions of lemma A.1,

$$\begin{pmatrix} \frac{dq^A}{dx} \\ \frac{dq^B}{dx} \end{pmatrix} = \begin{bmatrix} q_x^A + q_\alpha^A f_2^1 q_x^B - q_\beta^B f_1^2 q_x^A \\ q_x^B - q_\alpha^A f_1^1 q_x^B - q_\beta^B f_2^2 q_x^A \end{bmatrix}$$

where $x \varepsilon \{\Theta_A, \Theta_B\}$. Further, if $q^A = q^B$, then

$$\frac{dq^j}{dx} = q^j_x(1 - f_1 \cdot q^j_{\alpha j})$$

where $j = A, B, \alpha^A = \alpha, \beta^B = \beta$.

[*Note*: $f_1 < 0$ when $q^A = q^B$ from next lemma and $q^j_{\alpha j} \geq 0$ implies $\frac{dq^j}{dx} = 0$ *iff* $q^j_x = o$ whenever $q^A = q^B$.]

Lemma A.3. For

$$f(x,y) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!}{k!k!n-k!} x^k y^k (1-x-y)^{n-2k}$$

$$+ \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{n!}{k!k+1!(n-2k-1)!} x^k y^{k+1} (1-x-y)^{n-2k-1},$$

$$f_x = (y-x) \sum_{k=1}^{\left[\frac{n-1}{2}\right]} \frac{n!}{k+1!k-1!n-2k-1!} x^{k-1} y^k (1-x-y)^{n-2k-1}$$

$$- x \sum_{k=1}^{\left[\frac{n-1}{2}\right]} \frac{n!}{k!k+1!n-2k!} x^{k-1} y^k (1-x-y)^{n-2k-1}$$

$$- n(1-x-y)^{n-1},$$

$$f_y = (x-y) \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{n!}{k!k-1!n-2k!} x^{k-1} y^{k-1} (1-x-y)^{n-2k}.$$

Thus, for example, if $x=y$, then $f_y = 0$ and $f_x < 0$; if $x > y$, then $f_x < 0, f_y > 0$; and if $x < y$, then $f_y < 0$ and f_x is of indeterminate sign.

Lemma A.4. $(\alpha - \beta) = (q^B - q^A)\,\Gamma(q^A, q^B)$ where $\Gamma(q^A, q^B) \geq 0$.

Proof. $\alpha - \beta = f(q^A, q^B) - f(q^B, q^A) = (y-x)$

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{n!}{k!k+1!n-2k-1!} x^k y^y (1-x-y)^{n-2k-1}.$$

NOTES

I wish to thank both the Fairchild Foundation (as a Fairchild Scholar at Caltech) and the National Science Foundation (Grant #SOC 76-20953 to the Center for Mathematical Studies in Economics and Management Science, Northwestern University) for their support. My debt to John Ferejohn and Morris Fiorina, who tolerated my incursion into their domain with patience, help, and humor, will be obvious to any reader. A seminar at Caltech brought forth the expected boos, "What is that?" and help. I thank the participants. Finally, I alone am responsible for any errors and misrepresentations which may be left.

1. For a good summary of this literature, see Ferejohn and Fiorina (1974).
2. The exceptions include the model of Hinich, Ledyard, and Ordeshook (1972) in which voters choose probabilistically across voters. However, no model of individual decisions was given there to justify this behavior. Maybe none exists.

3. We use Kramer's (1977) terminology.
4. In their (1974) article (p. 527), they acknowledge the interactions of voters' decisions but suggest it is "a highly complex situation."
5. The basis for this claim is provided in section IIB(iii).
6. See the appendix for these facts.
7. See the article by Aranson, Hinich, and Ordeshook (1974) for these.
8. We later give an example of asymmetrically distributed tastes in which an electoral equilibrium does not exist.
9. The tail wags the dog?
10. For negative values of x, the proportion of voters with $d \leq x$ is $\frac{a}{a+1} e^x$; for positive values of x, the proportion of voters with $d \geq x$ is $\frac{1}{a+1} e^{-ax}$. For arbitrary $\bar{x} > 0$, $\frac{a}{a+1} e^{-\bar{x}} > \frac{1}{a+1} e^{-a\bar{x}}$ if $a > 1$.
11. See the appendix for the following fact.
12. See Hinich (1977) for a simliar result in a slightly different model.
13. Closeness may not count but it seems to be inevitable.
14. This approach must be well known to social-choice theorists. I welcome references on related work by others.
15. If voters' beliefs, and therefore their voting, are not independently and identically distributed, this approximation may be incorrect.
16. If $q^A > q^B$, then p_A is almost 1. If $q^A = q^B$, then p_A is almost $\frac{1}{2}$, and if $q^A < q^B$, then p_A is approximately 0.
17. See the appendix for this.
18. See Ferejohn and Fiorina (1974) for the case when $m = 3$.

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